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Some remarks on Dedekind lattices

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Abstract In this paper, we prove that a principally generated C-lattice L is a Dedekind lattice if and only if L is a WI-lattice in which every invertible element is a finite meet of powers of prime elements.

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الملخص

نثبت في هذه الورقة أن أي شبكة-C مولدة بشكل رئيس L هي شبكة ديديكند إذا كانت وفقط إذا كانت L شبكة-WI حيث يكون كل عنصر قابل للقلب التقاءً لعدد منتهٍ من قوى عناصر أولية.

1 Introduction

By a C-lattice L we mean a not necessarily modular complete multiplicative lattice ($a(\vee x_i) = \vee a x_i$) generated under joins by a multiplicatively closed subset C of compact elements, with least element 0 and compact greatest element 1, operating as the multiplicative identity. In any C-lattice multiplication defines a quotient operation by $a : b = \vee\{x \in L \mid xb \leq a\}$. Obviously C-lattices arise as abstractions of ideal systems, in particular when considering rings with identity. There the principal ideals form a generating set of compact "elements" whereas the finitely generated ideals form the set of all compact elements.

The theory of C-lattices was initiated by Dilworth in his fundamental and ground breaking paper [6] based on the notion of a principal element e. Recall that an element $e \in L$ is said to be principal if it satisfies:

$$(MP) \quad a \wedge be = ((a : e) \wedge b)e$$
$$(JP) \quad (ae \vee b) : e = (b : e) \vee a$$

In case that (MP) is satisfied, e is called "meet principal"; in case that (JP) is satisfied, e is called "join principal". If e satisfies (MP) only for $b = 1$, that is $a \wedge e = (a : e)e$ for all $a \in L$, then e is called "weak meet principal". Finite products of meet (join) principal elements are again meet (join) principal [6, Lemmas 3.3 and 3.4]. Moreover in [2, Theorem 1.3], it is shown that principal elements are always compact. For more information on principal elements, the reader is referred to [5].

Throughout this paper L denotes a principally generated C-lattice. For the definitions of prime element, maximal element, minimal prime element, and primary element, the reader is referred to [1,7]. An element $a \in L$ is called a nonzero divisor if $(0 : a) = 0$ and a is called invertible if a is a principal nonzero divisor. An element $a \in L$ is called regular if it contains an invertible element and a is called nilpotent if $a^n = 0$ for

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some positive integer n. If 0 is the only nilpotent element, then L is called reduced. For any $a, b \in L$, we say a and b are comaximal, if $a \vee b = 1$.

C-lattices can be localized. For any prime element p of L, L_p denotes the localization of L at $F = \{x \epsilon C \mid x \nleq p\}$. For details on C-lattices and their localization theory, the reader is referred to [7,12].

L is called a Prüfer lattice, if every compact element is principal. L is called a WI-lattice if every compact element $a \in L$ is principal and $(0 : (0 : a)) \vee (0 : a) = 1$. Note that by definition, $(0 : (0 : a)).(0 : a) = 0$. Prüfer lattices have been studied in [2,10]. A reduced lattice L is called quasi-regular, if for any compact element x, there is a compact element y such that $(0 : (0 : x)) = (0 : y)$. Quasi-regular lattices have been studied in [3]. Note that by [8, Theorem 4], L is a WI-lattice if and only if L is a quasi-regular lattice whose compact elements are principal. A reduced lattice L is called a Dedekind lattice if every element not contained in any minimal prime is "weak meet principal". For various characterizations of WI-lattices and Dedekind lattices, the reader is referred to [8,9,11].

It is well known that L is a Dedekind lattice if and only if L is a WI-lattice in which every invertible element is a finite product of prime elements [11, Theorems 2.6 and 3.12]. In this paper we prove that L is a Dedekind lattice if and only if L is a WI-lattice in which every invertible element is a finite meet of powers of prime elements. For general background and terminology, the reader may consult [1,2].

2 Nonminimal prime elements in WI-lattices

In this section we study nonminimal prime elements in WI-lattices in which every invertible element is a finite meet of powers of prime elements. Using these results, we establish that L is a Dedekind lattice if and only if L is a WI-lattice in which every invertible element is a finite meet of powers of prime elements.

We now prove some useful lemmas. It is well known that if L is a reduced lattice, then L is a Dedekind lattice if and only if every nonminimal prime is invertible [9, Theorem 9]. The following Lemma 2.1 shows that in a WI-lattice, every nonminimal prime element is the join of invertible elements.

Lemma 2.1 *Let L be a WI-lattice. Then every nonminimal prime element of L is the join of invertible elements.*

Proof Let p be a nonminimal prime element of L. As L is quasi-regular, by [3, Theorem 2], there exists a compact element $x \leq p$ such that $(0 : x) = 0$. As L is a WI-lattice, x is principal, so x is invertible, and hence p is regular. Let $p_r = \vee\{y \in L \mid y \leq p \text{ and } y \text{ is invertible}\}$. Clearly, $p_r \leq p$. Suppose $p_r < p$. Choose any principal element $a \leq p$ such that $a \nleq p_r$. As L is a WI-lattice, it follows that $x \vee a$ is invertible, so $x \vee a \leq p_r$, a contradiction. Therefore $p = p_r$ and hence every nonminimal prime element of L is the join of invertible elements. This completes the proof of the lemma. □

Lemma 2.2 *Let L be a WI-lattice in which every invertible element is a finite meet of powers of prime elements. Let m be a nonidempotent, nonminimal prime element of L. Then*

(i) *m is minimal over an invertible element of L.*
(ii) *m_m is invertible in L_m.*

Proof (i) Since $m \neq m^2$, by Lemma 2.1, there exists an invertible element $a \leq m$ such that $a \nleq m^2$. Choose any principal element $y \leq m$. As L is a WI-lattice, $a \vee y^2$ is invertible, so by hypothesis, $a \vee y^2 = \wedge_{i=1}^{n} {p_i}^{\alpha_i}$, where p_i's are prime elements of L. As $a \nleq m^2$, it follows that $\alpha_i = 1$ for all $p_i \leq m$. Again $(a \vee y^2)_m = \wedge\{(p_i)_m \mid p_i \leq m\} = (a \vee y)_m$, so by Nakayama's lemma (see [1, Theorem 1.1] or [2, Theorem 1.4]), $y_m \leq a_m$ and hence $m_m = a_m$. Therefore m is minimal over an invertible element a of L.

(ii) Again since $m_m = a_m$ and $(0 : a) = 0$, it follows that $(0_m : m_m) = (0_m : a_m) = (0 : a)_m = 0_m$, so m_m is invertible in L_m. □

Lemma 2.3 *Let L be a WI-lattice in which every invertible element is a finite meet of powers of prime elements. Let p be a nonminimal prime which is minimal over an invertible element $y \in L$. Then p^n is p-primary for all positive integers n.*

Proof Let n be a positive integer and let $r, s \in L$ be principal elements such that $rs \leq p^n$ and $s \nleq p$. Since y^n is invertible, by hypothesis, $r \vee y^n = \wedge_{i=1}^{m} {p_i}^{\alpha_i}$, where p_i's are prime elements of L. Since p is minimal over $r \vee y^n$, it follows that $(r \vee y^n)_p = (rs \vee y^n)_p = ({p_j}^{\alpha_j})_p$ where $p = p_j$ for some $j \in \{1, 2, \ldots, m\}$. But $(p^{\alpha_j})_p \leq (p^n)_p$ since $rs \vee y^n \leq p^n$, so $\alpha_j \geq n$, therefore $p^{\alpha_j} \leq p^n$ and hence $r \leq p^n$. This shows that p^n is p-primary for all positive integers n. This completes the proof of the lemma. □

Lemma 2.4 *Let L be a WI-lattice in which every invertible element is a finite meet of powers of prime elements. Let p be a nonidempotent, nonminimal prime element of L. Then*

(i) *$\{p^n\}_{n=1}^{\infty}$ is the set of all p-primary elements of L.*
(ii) *$p^{\omega} = \wedge_{n=1}^{\infty} p_n$ is a prime element of L.*
(iii) *If $q < p$ is a prime element of L, then $q \leq p^{\omega}$.*

Proof (i) Note that by Lemmas 2.2 and 2.3, $p^n \neq p^{n+1}$ for all positive integers n and p^n is p-primary for all positive integers n. Suppose q is p-primary. Then by [4, Lemma 3.2 (d)], $q = (p^n)_p = p^n$, so (i) holds.

(ii) Since p_p is invertible in L_p, by [4, Lemma 3.2 (c)], $p^{(\omega)} = \bar{\wedge}_{n=1}^{\infty}(p^n)_p$ ($\bar{\wedge}$ is the meet in L_p) is a prime element of L_p. It can be easily verified that p^{ω} is a prime element of L.

(iii) Follows from [4, Lemma 3.2 (c)]. □

Lemma 2.5 *Let L be a WI-lattice in which every invertible element is a finite meet of powers of prime elements. Then every invertible element is a finite meet of primary elements.*

Proof The proof of the lemma follows from Lemma 2.4 and [3, Lemma 8]. □

Definition 2.6 A regular prime element p of L is said to be a minimal regular prime if for any prime $q < p$, q is a nonregular prime element of L.

Lemma 2.7 *Let L be a WI-lattice in which every invertible element is a finite meet of powers of prime elements. If p is a nonidempotent, nonminimal prime element of L, then p is a minimal regular prime element of L.*

Proof Let p be a nonidempotent, nonminimal prime element of L and let $q < p$ be a prime element of L. Assume that q is a regular prime element of L. Suppose $b \leq q$ and $(0 : b) = 0$ for some principal element $b \in L$. Choose an invertible element $a \leq p$ such that p is minimal over a. Since ab is invertible, by Lemma 2.5, ab is a finite meet of primary elements of L. Let $ab = \wedge_{i=1}^{n} q_i$ be a normal primary decomposition of L. Let $q_i \leq p$ for $i = 1, 2, \ldots, k$ and $q_j \not\leq p$ for $j = k+1, \ldots, n$. Then $(ab)_p = \wedge_{i=1}^{k}(q_i)_p$. By Lemma 2.4, we can assume that $\sqrt{q_i} \leq p^{\omega}$ for $i = 1, 2, \ldots, k$. Then $a \not\leq \sqrt{q_i}$ for $i = 1, 2, \ldots, k$, so $b \leq \wedge_{i=1}^{k} q_i$ and hence $a_p b_p = b_p$. Therefore, by Nakayama's lemma, $b_p = 0_p$, a contradiction since $(0 : b) = 0$. This shows that p is a minimal regular prime element of L. This completes the proof of the lemma. □

Lemma 2.8 *Let L be a WI-lattice in which every invertible element is a finite meet of powers of prime elements. Suppose p is a prime minimal over an invertible element y of L. Then $p \neq p^2$.*

Proof If $p = p^2$, then by hypothesis, $p_p = y_p$, so by Nakayama's lemma $p_p = 0_p$, hence $y_p = 0_p$, a contradiction, since $(0 : y) = 0$. This shows that $p \neq p^2$. □

Lemma 2.9 *Let L be a WI-lattice in which every invertible element is a finite meet of powers of prime elements. If p is an idempotent prime, then p is a minimal prime element of L.*

Proof Suppose p is an idempotent prime element of L. Assume that p is nonminimal. Then there exists an invertible element $x \leq p$. By hypothesis, x has only finitely many minimal primes, say $p_1, p_2, \ldots, p_n$. By Lemma 2.8, $p \not\leq p_i$ for all i. As L is a WI-lattice, there exists a principal element $y \leq p$ such that $y \not\leq p_i$ for all i. Let $q \leq p$ be a prime minimal over $x \vee y$. If $q = q^2$, then by hypothesis, $(x \vee y)_q = q_q$, so by Nakayama's lemma $q_q = 0_q$ and therefore q is minimal, so by [3, Lemma 8], $x \not\leq q$, a contradiction. Therefore $q \neq q^2$ and nonminimal. By hypothesis and Lemma 2.7, q is a minimal regular prime. Again since $x \leq q$, it follows that $p_i < q$ for some i. This contradicts the fact that q is a minimal regular prime. Therefore p is a minimal prime element of L. □

Theorem 2.10 *L is a Dedekind lattice if and only if L is a WI-lattice in which every invertible element is a finite meet of powers of prime elements.*

Proof If L is a Dedekind lattice, then by [11, Theorem 2.6 (viii) and Theorem 3.12], L is a WI-lattice in which every invertible element is a finite meet of powers of prime elements. Conversely, assume that L is a WI-lattice in which every invertible element is a finite meet of powers of prime elements. We claim that every invertible element is a finite product of maximal prime elements. Let $a \in L$ be an invertible element and let $a = \wedge_{i=1}^{n} {p_i}^{\alpha_i}$, where ${p_i}'$s are distinct prime elements of L. Note that by [3, Theorem 2], ${p_i}'$s are nonminimal prime elements of L. Again by Lemmas 2.7 and 2.9, each p_i is maximal and so they are pairwise comaximal. Consequently, a is a finite product of maximal prime elements. Now the result follows from [11, Theorems 2.6 (viii) and 3.12]. This completes the proof of the theorem. □

References

1. Alarcon, F.; Anderson, D.D.; Jayaram, C.: Some results on abstract commutative ideal theory. Periodica Mathematica Hungarica **30**, 1–26 (1995)
2. Anderson, D.D.: Abstract commutative ideal theory without chain condition. Algebra Universalis **6**, 131–145 (1976)
3. Anderson, D.D.; Jayaram, C.; Phiri, P.A.: Baer lattices. Acta. Sci. Math. (Szeged) **59**, 61–74 (1994)
4. Anderson, D.D.; Jayaram, C.: Principal element lattices. Czechoslov. Math. J. **46**, 99–109 (1996)
5. Anderson, D.D.; Johnson, E.W.: Dilworth's principal elements. Algebra Universalis **36**, 392–404 (1996)
6. Dilworth, R.P.: Abstract commutative ideal theory. Pac. J. Math. **12**, 481–498 (1962)
7. Jayaram, C.; Johnson, E.W.: s-Prime elements in multiplicative lattices. Periodica Mathematica Hungarica **31**, 201–208 (1995)
8. Jayaram, C.; Johnson, E.W.: Dedekind lattices. Acta. Sci. Math. (Szeged) **63**, 367–378 (1997)
9. Jayaram, C.: ℓ-prime elements in multiplicative lattices. Algebra Universalis **48**, 117–127 (2002)
10. Jayaram, C.: Primary elements in Prüfer lattices. Czechoslov. Math. J. **52**(127), 585–593 (2002)
11. Jayaram, C.: Regular elements in multiplicative lattices. Algebra Universalis **59**, 73–84 (2008)
12. Johnson, J.A.; Sherette, G.R.: Structural properties of a new class of CM-lattices. Can. J. Math. **XXXVIII**, 552–562 (1986)